IMM-NYU 317
FEBRUARY 1964

NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

An Approximation Theorem

LIPMAN BERS

PREPARED UNDER
CONTRACT NO. NONR-285(46)
WITH THE
OFFICE OF NAVAL RESEARCH

IMM-NYU 317
February 1964

New York University

Courant Institute of Mathematical Sciences

AN APPROXIMATION THEOREM

Lipman Bers

This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Naval Research, Contract No. Nonr-285(46).

AN APPROXIMATION THEOREM*

Lipman Bers

The simple approximation theorem stated below, an incidental by product of an investigation with a different aim, seems not to be recorded in the literature. The proof uses a device due to Ahlfors.

Definition. Let D be a domain in the complex plane, $\dot{D}$ its boundary and $\Lambda \subset \dot{D}$ a closed set. We call Λ ample if (i) it contains every point of $\dot{D}$ which is not a boundary point of the complement G of $D \cup \dot{D}$, and (ii) in every component of G, the part of Λ contained in its boundary has positive harmonic measure.

Examples. Let D be the complement of a nowhere dense set Λ; then Λ is ample. Let D be bounded by k closed Jordan curves C_j and let λ_j be a subarc of Λ_j; then $\Lambda = \lambda_1 \cup \ldots \cup \lambda_k$ is ample. If C_j is rectifiable, it suffices to assume that $\lambda_j \subset C_j$ has positive linear measure.

Theorem. Let Λ be a set on the boundary of a plane domain D and assume that the closure of Λ is ample. Let f(z) be analytic in D and such that

$$(1) \qquad \iint_D |f(z)| \, dx \, dy < +\infty.$$

*This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Naval Research, Contract No. Nonr-2 [illegible].

Then there exists a sequence of rational functions $r_j(z)$, _with simple poles in_ Λ _and no other singularities, such that_

(2) $$\lim_{j\to\infty} \iint_D |f(z) - r_j(z)|\, dx\, dy = 0.$$

Proof. We assume Λ to be infinite; otherwise the statement is trivial. Let α denote the set of rational functions with simple poles in Λ, which are absolutely integrable over D. Analytic functions satisfying (1) form a Banach space. Let ℓ be a continuous linear functional on this space. It suffices to show that if $\ell(\phi) = 0$ for all ϕ in α, then $\ell \equiv 0$.

Every ℓ is of the form

(3) $$\ell(f) = \iint_D f(z)\,\mu(z)\, dx\, dy$$

where μ is a bounded measurable function. Let a_1 and a_2 be two points in Λ and set

(4) $$h(z) = -\frac{(z - a_1)(z - a_2)}{\pi} \iint_D \frac{\mu(\zeta)\,d\xi\, d\eta}{(\zeta - z)(\zeta - a_1)(\zeta - a_2)}$$

Then h(z) is continuous everywhere, $h(a_1) = h(a_2) = 0$, h has generalized derivatives which are locally square integrable,

(5) $$\partial h/\partial \bar{z} = \mu \text{ in D}$$

and h(z) is analytic in the complement G of the closure of D. Also,

(6) $$h(z) = O(|z| \log |z|), \qquad z \to \infty,$$

and, for every $R > 0$,

(7) $$|h(z') - h(z'')| \leqslant C(R)\, |z' - z''| \; |\log |z' - z''|| \qquad \text{for } |z'|, |z''| \leqslant R.$$

All this is verified by standard arguments.

Assume that $\ell(\phi) = 0$ for all φ in α. For every $a \in \Lambda$, $a \neq a_1, a_2$, the function $\phi(\zeta) = (\zeta - a)^{-1}(\zeta - a_1)^{-1}(\zeta - a_2)^{-1}$ belongs to α. For this ϕ, $-\pi h(a) = (a - a_1)(a - a_2)\ell(\phi)$. Thus $h = 0$ on the closure of Λ. Using condition (ii) of ampleness we conclude that $h \equiv 0$ in G, and hence

(8) $$h = 0 \text{ on } \dot{D}$$

Let $\delta(z)$ denote the distance from z to $\dot{D}$; by (6) and (8)

(9) $$|h(z)| \leqslant C(R)\, \delta(z)\, |\log \delta(z)| \text{ for } |z| \leqslant R.$$

Now let $j(t)$, $-\infty$ be an infinitely differentiable function such that $0 < j(t) < 1$, $j(t) = 0$ for $t \leqslant 1$, $j(t) = 1$ for $t > 1$ and set, for $n = 1, 2, \ldots$, and for z in D,

$$\omega_n(z) = j\left(-n / \log\log \frac{1}{\delta(z)} \right)$$

(this device is due to Ahlfors). Since $\delta(z)$ is Lipschitz continuous with constant 1, and $j'(t) = 0$ outside the interval $1 < t < 2$, one verifies that

(10) $$\left| \frac{\partial \omega_n(z)}{\partial z} \right| \leqslant \frac{c}{n} \; \frac{}{\delta(z)\, |\log \delta(z)|}.$$

For every $R > 0$, let $D(R)$ and $\Gamma(R)$ denote the intersection of D with the disc $|z| < R$ and the circle $|z| = R$, respectively. By (8) and Stokes' theorem

$$\iint\limits_{D(R)} \omega_n(z) f(z) \mu(z)\, dx\, dy = \iint\limits_{D(R)} \omega_n(z) \frac{\partial}{\partial \bar z} (h(z) f(z))\, dx\, dy$$

$$= -\frac{i}{2} \int\limits_{\Gamma(R)} \omega_n(z) h(z) f(z) dz - \iint\limits_{D(R)} f(z) h(z) \frac{\partial \omega_n(z)}{\partial \bar z}\, dx\, dy$$

for every $f(z)$ analytic in D, since $\omega_n \equiv 0$ near $\dot D$. Assume now that (1) holds. By (9) and (10) the last integral goes to 0 for $n \to \infty$, and, since $\omega_n \to 1$, we conclude that

$$\left| \iint\limits_{D(R)} f(z) \mu(z)\, dx\, dy \right| \leq \left| \frac{1}{2} \int\limits_{\Gamma(R)} f(z) h(z)\, dz \right|.$$

Here the right hand side vanishes for large R if D is bounded, but is in all cases less than

$$\text{(11)} \qquad \text{const. } R \log R \int\limits_{\Gamma(R)} |f(z)|\, |dz| \qquad (R > 1)$$

in view of (7), Since

$$\int_0^{+\infty} \left\{ \int\limits_{\Gamma(R)} |f(z)|\, |dz| \right\} dR < +\infty$$

by (1), the quantity (11) can not remain above a positive number as $R \to \infty$. We conclude from (3) that $\ell(f) = 0$, q.e.d.

OFFICIAL DISTRIBUTION LIST

Chief of Naval Research
Navy Department
Washington 25, D.C.
Attn: Code 432 — 5
Code 438 — 3

Director
Naval Research Laboratory
Washington 25, D.C.
Attn: Library — 2
Code 6230 — 2
Tech. Information Officer — 6

Commanding Officer
Office of Naval Research
Branch Office
207 West 24th Street
New York 11, New York

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California
Attn: Tech. Library

Commanding Officer
Office of Naval Research
Branch Office
495 Summer Street
Boston 10, Massachusetts

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco, California

Commanding Officer
Office of Naval Research
Branch Office
Navy No. 100, Fleet Post Office
New York, New York — 40

Chief, Bureau of Ships
Navy Department
Washington 25, D.C.
Attn: Library — 2
Code 280 — 1

Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
Attn: Tech. Library

Director
David Taylor Model Basin
Washington 25, D.C.
Attn: Library — 1
Dr. H. Polachek — 1

U.S. Naval Electronics Laboratory
San Diego 52, California
Attn: Library — 1
Dr. F. A. Sabransky — 1

U.S. Naval Weapons Plant
Washington 25, D.C.
Attn: Library

U.S. Navy Unerwater Sound Lab.
Fort Trumbull
New London, Connecticut

Naval Ordnance Laboratory
White Oak
Silver Spring, Maryland
Attn: Mechanics Division — 1
Library — 1

U.S. Naval Hydrographic Office
Suitland, Maryland — 2

Beach Erosion Board
U.S. Corps of Engineers
Little Falls Road, N.W.
Washington 16, D.C.

Superintendent
U.S. Naval Postgraduate School
Monterey, California

Commander
U.S. Naval Weapons Laboratory
Dahlgren, Virginia
Attn: Library

Commanding General
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Library

Commanding General
Wright-Patterson Air Force Base, Ohio
Attn: Central Air Documents (D13) 1
Aeronautical Research Lab. 1

Chief
Armed Forces Special Weapons Project
Washington 25, D.C.

Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia 10

U. S. Department of Commerce
Washington 25, D.C.
Attn: National Hydraulics Lab.

The Computing Laboratory
National Applied Mathematics Lab.
National Bureau of Standards
Washington 25, D.C.

California Institute of Technology
Hydrodynamics Laboratory
Pasadena, California

University of California
Department of Engineering
Berkeley 4, California
Attn: Dr. J.W. Johnson 1
Dr. S. Schaaf 1

Carnegie Institute of Technology
Department of Mathematics
Pittsburgh, Pennsylvania

Chesapeake Bay Institute
The John Hopkins University
121 Maryland Hall
Baltimore 18, Maryland
Attn: Director,
Dr. D. W. Pritchard

University of Chicago
Department of Meteorology
Chicago, Illinois
Attn: Dr. C.G. Rossby

Columbia University
New York 27, New York
Attn: Prof. M.G. Salvadori

Harvard University
Department of Mathematics
Cambridge, Massachusetts
Attn: Prof. G. Birkhoff

Indiana University
Department of Mathematics
Bloomington, Indiana
Attn: Prof. T.Y. Thomas

State University of Iowa
Iowa Institute of Hydraulic Research
Iowa City, Iowa
Attn: Prof. L. Landweber

Massachusetts Institute of Technology
Cambridge 38, Massachusetts
Attn: Dr. E. Reissner 1
Dr. C. C. Lin 1

New York University
Department of Meteorology
New York, New York
Attn: Dr. W. J. Pierson, Jr.

Princeton University
Department of Mathematics
Princeton, New Jersey
Attn: Prof. S. Lefschetz

Rand Corporation
1700 Main Street
Santa Monica, California

Scripps Institute of Oceanography
La Jolla, California
Attn: Dr. W. Munk 1
Dr. R.S. Arthur 1

University of Washington
Oceanographic Department
Seattle 5, Washington
Attn: Dr. Maurice Rattray, Jr. 1

Woods Hole Oceanographic Institute
Woods Hole, Massachusetts
Attn: Dr. C. Iselin

Marine Biological Laboratory
Woods Hole, Massachusetts
Attn: Library

Dr. Milton Rose
Radiation Laboratory
Livermore, California

Dr. Arthur Grad
National Science Foundation
Washington 25, D.C.

Serials Work Room
Main Library, 206
New York University
New York, New York

Dr. C. R. DePrima
California Institute of Technology
1201 East California Street
Pasadena, California

Office of Technical Services
Department of Commerce
Washington 25, D. C.

Department of Mathematics
Stanford University
Stanford, California

Department of Mathematics
Harvard University
Cambridge, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Professor B. Friedman
Department of Mathematics
University of California
Berkeley, California

Director (Code 120)
USN Underwater Sound Reference Lab.
P.O. Box 8337
Orlando, Florida

Professor A. Erdelyi
Department of Mathematics
California Institute of Technology
Pasadena, California

Professor J. Todd
Department of Mathematics
California Institute of Technology
Pasadena, California

Professor C. H. Wilcox
Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Professor B. Zumino
Department of Physics
New York University
New York, New York

Department of Mathematics
University of California
Berkeley, California

Professor H. C. Kranzer
Department of Mathematics
Adelphi College
Garden City, New York

Dr. F. J. Weyl
Research Director, Code 402
Office of Naval Research
Washington 25, D. C.

DATE DUE

APR 13 '64			
GAYLORD			PRINTED IN U.S.A.

N.Y.U. Courant Institute of
Mathematical Sciences
4 Washington Place
New York 3, N. Y.